SCATTERING THEORY AND DEFORMATIONS OF ASYMPTOTICALLY HYPERBOLIC METRICS

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Abstract. For an asymptotically hyperbolic metric on the interior of a compact manifold with boundary, we prove that the resolvent and scattering operators are continuous functions of the metric in the appropriate topologies.

Contents

1. Introduction

Let X be a compact manifold with boundary and g an asymptotically hyperbolic metric on the interior of X, in the sense that all sectional curvatures approach -1 at ∂X. We will assume that g is of the form

$$g = \rho^{-2} h,$$

where ρ is a boundary defining function on X and h is a Riemannian metric on X. Such a metric g is necessarily complete. The asymptotic curvature condition reduces to $|d\rho|_h = 1$ on ∂X. Any compact manifold with boundary possesses such a metric. The chief examples are purely hyperbolic, with the interior of X isometric to $\mathbb{H}^{n+1}$ or a convex co-compact quotient $\mathbb{H}^{n+1}/\Gamma$.

Let Δ_g be the Laplacian associated to g, acting on functions. It was proven in [8], [12] that the spectrum of Δ_g consists of absolutely continuous spectrum $[n^2/4, \infty)$, plus finite point spectrum $\mathrm{Spec}(\Delta_g) \subset (0, n^2/4)$ (no embedded eigenvalues, in particular).

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In [14] Mazzeo and Melrose demonstrated the meromorphic continuation of the (modified) resolvent,

$$R_\zeta := [\Delta_g - \zeta(n-\zeta)]^{-1}, \qquad \dim X = n+1.$$

The proof involves the careful construction of a parametrix for $\Delta_g - \zeta(n-\zeta)$. This construction is part of the general program for dealing with degenerate elliptic boundary problems originated by Melrose and to be presented in detail in [20].

The continuous spectrum of Δ_g may be characterized by the behavior of generalized eigenfunctions at infinity. If $\operatorname{Re}\zeta = n/2$, $\zeta \neq n/2$, then for each $f \in C^\infty(\partial X)$ there exists a unique solution of $\Delta_g u = \zeta(n-\zeta)u$ with asymptotic behavior

$$u = \rho^{n-\zeta} f + \rho^\zeta f' + O(\rho^{n/2+\epsilon}),$$

where $f' \in C^\infty(\partial X)$, $\epsilon > 0$. In fact such a solution will have a complete polyhomogeneous expansion in ρ [13].

The "scattering operator," defined by

$$S_\zeta : f \mapsto f', \tag{1.1}$$

is a zeroth-order pseudodifferential operator on ∂X. This follows fairly directly from the results of [14], although the scattering operator was not considered there. [1] For the case $X = \mathbb{H}^{n+1}/\Gamma$, the meromorphic continuation of the resolvent was also proven by Perry [22] using a method involving the scattering operator, which was shown to be pseudodifferential explicitly.

As defined by (1.1), S_ζ depends on the choice of boundary defining function ρ. To remove this dependency, we may introduce the bundles Ω^α of α-densities on ∂X, where $\alpha \in \mathbb{C}$. Let γ_h be the density on ∂X coming from the metric induced by h. Then we define the normalized scattering operator

$$\check{S}_\zeta : C^\infty(\partial X; \Omega^{1-\zeta/n}) \to C^\infty(\partial X, \Omega^{\zeta/n}),$$

by

$$\check{S}_\zeta : f \cdot (\gamma_h)^{1-\zeta/n} \mapsto (S_\zeta f)\cdot(\gamma_h)^{\zeta/n}.$$

It is easily seen that $\check{S}_\zeta$ depends only on g and not on ρ. For $X = \mathbb{H}^{n+1}/\Gamma$, $C^\infty(\partial X; \Omega^\alpha)$ may be conveniently realized as a space of automorphic forms on the regular set of Γ, and the kernel of $\check{S}_\zeta$ may be written as an average over Γ of the scattering kernel for $\mathbb{H}^{n+1}$. This coincides with the scattering operator as studied in [1], [2], [5], [6], [9], [21], [22], [23].

We will find it more convenient to deal with S_ζ rather than $\check{S}_\zeta$. Of course the results apply to either definition. The scattering operator also extends to a meromorphic family in ζ, and it may be derived from the resolvent by taking a certain limit at the boundary.

The question we will address is the continuity of the resolvent and scattering operator under deformations of the metric. Denote by $\mathcal{M}_X$ the space of asymptotically hyperbolic metrics on X, with the topology inherited from $\rho^{-2}C^\infty(X, T^*X \otimes T^*X)$. This is just the C^∞ topology on the metric h.

A refinement of the construction of [14], given in [13], shows that $\tilde{\mathcal{K}}_{R_\zeta}$, the lift of the Schwarz kernel of R_ζ to the stretched product $X \tilde{\times} X$, is a distribution with

[1] See [19] for a review of scattering theory on manifolds with various types of regular structure at infinity.

polyhomogeneous conormal singularities at the boundary. In the notation to be introduced in §3,

$$R_\zeta \in \tilde{\Psi}^{-2}_{\zeta,\zeta,0}(X;\, \Omega^{1/2}) + \Psi_{\zeta,\zeta}(X;\, \Omega^{1/2}).$$

Roughly this means that the lifted kernel has full asymptotic expansions near all boundaries, which are polyhomogeneous in the appropriate boundary defining function. The subscripts give the leading orders of these expansions. The topology of polyhomogeneous conormal distributions controls the behavior of all coefficients in the boundary expansions.

Theorem 1.1. *For $\zeta \neq n/2$, the map*

$$\mathcal{M}_X \ni g \mapsto R_\zeta \in \tilde{\Psi}^{-2}_{\zeta,\zeta,0}(X;\, \Omega^{1/2}) + \Psi_{\zeta,\zeta}(X;\, \Omega^{1/2})$$

is continuous except at poles.

The implications for the scattering operator are easier to describe. From the structure of the resolvent we may deduce the local form of the kernel:

$$K_{S_\zeta}(y, y') = r^{-2\zeta} F(r, \theta, y) + G(y, y'),$$

where (y, y') are local coordinates for $\partial X \times \partial X$, $r = |y - y'|$, $\theta = (y - y')/r$, and F and G are smooth in their respective variables. This implies that S_ζ is a pseudodifferential operator of order $2\zeta - n$ with a one-step polyhomogeneous symbol expansion. From Theorem 1.1 we will deduce that the maps $g \mapsto F, G$ are continuous in a C^∞ topology. If $\Psi^a(\partial X)$ is the space of all one-step polyhomogeneous pseudodifferential operators of order a, with the appropriate topology (defined in §4), then we have the following.

Theorem 1.2. *For $\zeta \neq n/2$, the map*

$$\mathcal{M}_X \ni g \mapsto S_\zeta \in \Psi^{2\zeta - n}(\partial X)$$

is continuous except at poles.

In joint work with Peter Perry, these results will be applied to study the behavior of scattering poles and resonances (poles of the resolvent) under metric deformations [3].

A converse to this result was proven in [2], in the special case where $X = \mathbb{H}^3/\Gamma$. There it was shown that the size of a quasiconformal deformation of Γ is controlled by the change in scattering operator, in the operator topology. The standard topology of quasiconformal deformations is, however, essentially a C^0 topology, i.e. much weaker than that of $\mathcal{M}_X$. In [4], a homeomorphism will be established between the quasiconformal deformation space of $\mathbb{H}^3/\Gamma$ and the space of scattering operators endowed with a suitably weak topology.

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2. The hyperbolic model

One of the key features of the analysis of [14] is the relation of the general case back the model case of the ball $\mathbb{B}^{n+1}$ with g the standard hyperbolic metric. It is generally more convenient to deal with the half-space model $\mathbb{H}^{n+1}$ with coordinates $z = (x, y)$, $x \geq 0$, and the metric $g = (dx^2 + dy^2)/x^2$.

The Laplacian in these coordinates is

$$\Delta = -(x\partial_x)^2 + nx\partial_x - \sum_{i=1}^{n} (x\partial_{y_i})^2.$$

Let $G_\zeta(z, z')$ be Schwartz kernel of the resolvent $(\Delta - \zeta(n-\zeta))^{-1}$, with respect to the measure dg. G_ζ is purely a function of the hyperbolic distance $d(z, z')$, given by

$$\cosh d(z, z') = 1 + \frac{|z - z'|^2}{2xx'}.$$

Explicitly,

$$G_\zeta(z, z') = c_\zeta \left(\cosh \frac{d}{2}\right)^{-2\zeta} F(\zeta, \zeta - \tfrac{n-1}{2}, 2\zeta - n + 1; (\cosh \tfrac{d}{2})^{-2}),$$

where F is the hypergeometric function

$$F(a, b, c; u) = 1 + \frac{a \cdot b}{1 \cdot c}\, u + \frac{a(a+1) \cdot b(b+1)}{1 \cdot 2 \cdot c(c+1)}\, u^2 + \dots,$$

and

$$c_\zeta = \pi^{-n/2} 2^{-2\zeta - 1} \frac{\Gamma(\zeta)}{\Gamma(\zeta - \frac{n}{2} + 1)}. \tag{2.1}$$

We can define a generalized eigenfunction for Δ by taking

$$E_\zeta(z, y') = \lim_{x' \to 0} (x')^{-\zeta} G_\zeta(z, z').$$

E_ζ is a smooth function of z in the interior, and an eigenfunction in the sense that

$$(\Delta - \zeta(n - \zeta)) E_\zeta(\cdot, y') = 0.$$

This follows immediately from $(\Delta - \zeta(n-\zeta)) G_\zeta(\cdot, z') = \delta_{z'}$. Or one can just check this explicitly, since

$$E_\zeta(z, y') = c_\zeta \left[\frac{x}{x^2 + |y - y'|^2}\right]^\zeta$$

Given $f \in C_c^\infty(\mathbb{R}^n)$, one can form a solution to $\Delta u = \zeta(n - \zeta)u$ by integrating this generalized eigenfunction

$$u(z) = 2^{2\zeta}(2\zeta - n) \int_{\mathbb{R}^n} E_\zeta(z, y') f(y')\, d^n y.$$

Proposition 2.1. *For* $\operatorname{Re}\zeta = n/2$, $\zeta \neq n/2$, u *has asymptotic behavior*

$$u(z) = x^{n-\zeta} f(y) + x^\zeta f'(y) + O(x^{n/2+1}),$$

near $x = 0$. *Moreover* $f' = S_\zeta f$, *where* S_ζ *is the zeroth order pseudodifferential operator on* $\mathbb{R}^n$ *with total symbol*

$$a(y, \xi) = 2^{n-2\zeta} \frac{\Gamma(\frac{n}{2} - \zeta)}{\Gamma(\zeta - \frac{n}{2})}\, |\xi|^{2\zeta - n}.$$

Proof. Define $W_x \in \mathcal{S}'(\mathbb{R}^n)$ by

$$W_x(|w|) = \frac{x^\zeta}{(|w|^2 + x^2)^\zeta}.$$

Then the partial Fourier transform of u is $\hat{u}(x,\xi) = c_\zeta \hat{W}_x(|\xi|)\hat{f}(\xi)$. The distributional Fourier transform $\hat{W}_1(|\xi|)$ is the analytic continuation of a Bessel function, so $\hat{W}_x$ may be analyzed with standard tricks (see for example Mandouvalos [10]). For ζ as above, as $x \to 0$ we have

$$\hat{W}_x(\xi) = \pi^{n/2}\frac{\Gamma(\zeta - \frac{n}{2})}{\Gamma(\zeta)}\, x^{n-\zeta} + \pi^{n/2}\frac{\Gamma(\frac{n}{2} - \zeta)}{\Gamma(\zeta)}\, x^\zeta \left(\frac{|\xi|}{2}\right)^{2\zeta - n} + O(x|\xi|)$$

□

3. Stretched products and distributions

The Laplacian Δ_g for an asymptotically hyperbolic metric may be regarded as an operator on X which is elliptic in the interior but degenerates uniformly at the boundary. It belongs to Diff_0^m, the enveloping algebra of the space of smooth vector fields on X which vanish at the boundary. In local coordinates $z = (x, y)$, with x a boundary defining function, such operators have the form

$$\sum_{|\alpha| \le m} a_\alpha(z)\big(x\frac{\partial}{\partial z}\big)^\alpha.$$

In order to even state our main result, we need to review the calculus of distributions used to analyze the inverses of such operators.

As usual, in order to gave a nice symbol map, we want to consider operators acting on half-densities. Over a manifold W let $\Omega^{1/2}(W)$ denote the bundle of half-densities. We will simply write $\Omega^{1/2}$ when the manifold is clear from context.

3.1. **Stretched product.**

Let $\dot{C}^\infty(X; \Omega^{1/2})$ be the space of smooth half-densities vanishing to infinite order on ∂X, and $\mathcal{D}'(X; \Omega^{1/2})$ the space of distributional half-densities extendible across ∂X. Linear continuous operators $\dot{C}^\infty(X; \Omega^{1/2}) \to \mathcal{D}'(X; \Omega^{1/2})$ have Schwartz kernels which are extendible distributional half-densities on $X \times X$. Note that $X \times X$ is a manifold with two boundary hypersurfaces, plus a corner $\partial X \times \partial X$ where they meet. A crucial element of the parametrix construction of [14] is the resolution of singularities of Schwartz kernels at the submanifold $\partial\Lambda$ where the diagonal $\Lambda \subset X \times X$ meets the corner. This is done using the technique of [15], the introduction of a "stretched product" $X \tilde{\times} X$. $X \tilde{\times} X$ is obtained from $X \times X$ by blowing up $\partial\Lambda$, which essentially means that for each point on $\partial\Lambda$ we keep track of a direction of approach from $X \times X$. The new manifold has three boundary hypersurfaces: The left and right faces F_f and F_r, corresponding to $\partial X \times X$ and $X \times \partial X$ in the original product, and the front face F_f, which is the replacement of $\partial\Lambda$.

Locally this is just the introduction of polar coordinates. Let (x, y) be coordinates in X, with x a defining function for ∂X. Then (x, y, x', y') give a set of coordinates for $X \times X$ near the boundary of Λ ($\partial\Lambda$ being given locally by $\{x = x' = 0,\ y = y'\}$). We introduce polar coordinates around $\partial\Lambda$:

$$r = \sqrt{x^2 + (x')^2 + |y - y'|^2}, \qquad (\eta, \eta', \theta) = \frac{(x, x', y - y')}{r}.$$

Here (η, η', θ) lives in a closed quarter sphere of S^{n+1}, since $\eta, \eta' \geq 0$. The full set of local coordinates for $X \tilde{\times} X$ is $(r, \eta, \eta', \theta, y)$, and the faces are locally given by

$$F_l = \{\eta = 0\}, \qquad F_r = \{\eta' = 0\}, \qquad F_f = \{r = 0\}.$$

The global description of the stretched product is as follows. We take $X \times X$, remove $\partial\Lambda$, and then in its place we glue the doubly inward pointing spherical normal bundle of $\partial\Lambda$. This is given a smooth structure using the polar coordinate patches introduced above. Clearly there is a smooth map $b : X \tilde{\times} X \to X \times X$ which collapses the blow up. In terms of this map,

$$F_l = b^{-1}(\partial X \times X), \quad F_r = b^{-1}(X \times \partial X), \quad F_f = b^{-1}(\partial X \times \partial X).$$

Let ϕ_l, ϕ_r, ϕ_f be fixed defining functions for the faces $F_l, F_r, F_f \subset X \tilde{\times} X$, respectively.

For many purposes it is convenient to use the projective coordinates (x, y, t, u), where $t = x/x'$ and $u = (y - y')/x'$. In these coordinates the left, right, and front faces are $t = 0$, $t = \infty$, and $x = 0$, respectively.

Given a linear continuous operator $A : \dot{C}^\infty(X; \Omega^{1/2}) \to \mathcal{D}'(X; \Omega^{1/2})$, let $\mathcal{K}_A$ be its Schwartz kernel. We will characterize such operators by the pull-back of the kernel to $X \tilde{\times} X$:

$$\tilde{\mathcal{K}}_A := b^* \mathcal{K}_A.$$

Let π_l and π_r be the maps $X \tilde{\times} X \to X$ which correspond to projections onto the left and right factors in the interior. Then given $\tilde{\mathcal{K}}_A$, we recover the action of A by

$$A\mu = \pi_{l*}(\tilde{\mathcal{K}}_A \pi_r^* \mu),$$

for $\mu \in C^\infty(X, \Omega^{1/2})$. To illustrate this in the local coordinates (x, y, t, u), suppose

$$\mathcal{K}_A = k(x, y, x', y') \left| \frac{dx\, dy\, dx'\, dy'}{(xx')^{n+1}} \right|^{1/2}.$$

Here we include the factors of x and x' in the denominator because that is the form of the Riemannian half-density μ_g. Then

$$\tilde{\mathcal{K}}_A = k(x, y, \tfrac{x}{t}, y - \tfrac{xu}{t}) \left| \frac{dx\, dy\, dt\, du}{t x^{n+1}} \right|^{1/2}.$$

If $\mu = f(x,y) |\frac{dx\, dy}{x^{n+1}}|^{1/2}$ then

$$\pi_r^* \mu = f(\tfrac{x}{t}, y - \tfrac{xu}{t}) \left| \frac{dt\, du}{t} \right|^{1/2}.$$

The product $\tilde{\mathcal{K}}_A \pi_r^* \mu$ is a density in the t and u variables and can be pushed forward to give

$$A\mu = \int k(x, y, \tfrac{x}{t}, y - \tfrac{xu}{t}) f(\tfrac{x}{t}, y - \tfrac{xu}{t}) \frac{dt\, du}{t} \cdot \left| \frac{dx\, dy}{x^{n+1}} \right|^{1/2}. \tag{3.1}$$

Finally we note that for the identity operator I,

$$\tilde{\mathcal{K}}_I = \delta(t-1)\delta(u) \left| \frac{dx\, dy\, dt\, du}{x^{n+1}} \right|^{1/2} \tag{3.2}$$

3.2. **Distributions.** The Riemannian half-density μ_g is a section of the singular density bundle $(\rho\rho')^{-(n+1)/2}\Omega^{1/2}(X\times X)$. The pull-back of this bundle to $X\tilde{\times}X$ is

$$b^*\Big[(\rho\rho')^{-(n+1)/2}\Omega^{1/2}(X\times X)\Big]=\tau\Omega^{1/2}(X\tilde{\times}X),$$

where

$$\tau=(\phi_l\phi_r\phi_f)^{-(n+1)/2}.$$

Accordingly, we will define spaces of operators whose kernels are sections of this bundle.

Let $\tilde{\Lambda}$ denote the closure of the lift to $X\tilde{\times}X$ of the interior of the diagonal in $X\times X$. Note that $\tilde{\Lambda}$ intersects the front face transversally. We define $I^m(X\tilde{\times}X,\tilde{\Lambda};\,\tau\Omega^{1/2})$ to be the space of distributional sections of $\tau\Omega^{1/2}(X\tilde{\times}X)$ which are conormal to $\tilde{\Lambda}$ of degree m as in [7] and which vanish to all orders at F_l and F_r. Define

$$\tilde{\Psi}^m(X;\,\Omega^{1/2}):=\{A:\dot{C}^\infty(X,\Omega^{1/2})\to\mathcal{D}'(X,\Omega^{1/2});\ \tilde{\mathcal{K}}_A\in I^m(X\tilde{\times}X,\tilde{\Lambda};\,\tau\Omega^{1/2})\}.$$

Because of the vanishing at the left and right faces, the factors of ϕ_l and ϕ_r in $\tau\Omega^{1/2}$ are irrelevant to the definition. To see the significance of the ϕ_f, note from from (3.2) that $\tilde{\mathcal{K}}_I$ is a smooth section of $\tau\Omega^{1/2}$ but would be singular as a section of $\Omega^{1/2}$ (in those coordinates $\phi_f=x$). So the definition above gives the desired fact that $I\in\tilde{\Psi}^0(X;\,\Omega^{1/2})$.

The topology of $I^m(X\tilde{\times}X,\tilde{\Lambda};\,\tau\Omega^{1/2})$ is defined as follows. Since the distributions are extendible it is convenient to consider $(X\tilde{\times}X)^2$, which is the double of $X\tilde{\times}X$ across this face. Near the diagonal of $(X\tilde{\times}X)^2$ we take local Fourier transforms and use the topology of the standard symbol spaces S^m. Away from the diagonal and the topology is that of $(\tilde{\phi}_l\tilde{\phi}_r)^\infty C^\infty((X\tilde{\times}X)^2)$, where $\tilde{\phi}_l$ and $\tilde{\phi}_r$ are any smooth extensions of ϕ_l and ϕ_r. The topology of $I^m(X\tilde{\times}X,\tilde{\Lambda};\,\tau\Omega^{1/2})$ is simply the restriction of this topology on $(X\tilde{\times}X)^2$. This definition makes $I^m(X\tilde{\times}X,\tilde{\Lambda};\,\tau\Omega^{1/2})$ a complete locally convex vector space.

It is important to note that operators in $\tilde{\Psi}^{-\infty}(X;\,\Omega^{1/2})$ have kernels which are smooth when pulled back to $X\tilde{\times}X$, but not necessarily smooth on $X\times X$.

Our constructions will also involve kernels with polyhomogeneous conormal singularities at the boundaries. We'll review the facts we need; see [16], [18], [20] for more complete expositions. Let W be a manifold with corners. Label the boundary faces $1,\dots,k$, with corresponding boundary defining functions ϕ_j. Polyhomogeneous conormal singularities are described by the powers of ϕ_j and $\log\phi_j$ which occur in the expansions at each face. For each $j=1,\dots,k$ an *index set* E_j, a countable discrete subset of $\mathbb{C}\times\mathbb{N}_0$. The collection $\mathcal{E}=\{E_1,\dots,E_k\}$ is called an *index family* for W. The space of *polyhomogeneous conormal distributions*[2] $\mathcal{A}_{\mathcal{E}}(W)$ consists of functions u which are smooth on the interior of W and which near the $j-$th boundary face have an asymptotic expansion of the form:

$$u\sim\sum_{(a,l)\in E_j}\ \sum_{0=1,\dots,l}\phi_j^a(\log\phi_j)^l u_{a,l},\tag{3.3}$$

[2]We omit the usual designation "phg" from the notation because all distributions in this paper are polyhomogeneous at boundaries.

where the $u_{a,l}$ are smooth. (To give a proper definition one must take somewhat more care with the corners; we refer the reader to [20].) To insure that this expansion makes sense, for each index set E it is required that the set

$$E \cap (\{\mathrm{Re}\, a < M\} \times \mathbb{N}_0)$$

is finite for all $M \in \mathbb{R}$. If u vanishes to infinite order at the $j-$th face, then we write $E_j = \infty$. The definitions are local and can be applied to sections of line bundles.

For simplicity, we will abbreviate $E = \{(a,0)\}$ simply as a. This indicates singularities of a very simple form:

$$\mathcal{A}_{a_1,\dots,a_k}(W) = \phi_1^{a_1} \dots \phi_k^{a_k}\, C^\infty(W).$$

The spaces of polyhomogeneous conormal distributions are given topologies as follows. We first define the C^∞ seminorms. Let $\mathcal{V}_b(W)$ be the set of smooth vector fields tangent to all boundary faces, and choose a set $\{V_1, \dots, V_l :\ V_j \in \mathcal{V}_b(W)\}$ which together span $\mathcal{V}_b(W)$ everywhere. The index sets have a partial ordering: $(a,l) < (b,k)$ if $\mathrm{Re}\, a < \mathrm{Re}\, b$ or if $a = b$ and $l > k$. So given an index family $\mathcal{E}$ we define $\mathfrak{e}$ by choosing a smallest member of each index set (this may not be unique). For $k \in \mathbb{N}_0$ we define the norm on $\mathcal{A}_{\mathcal{E}}(W)$:

$$\|u\|_{k;\mathfrak{e}} = \sum_{|\alpha| \leq k} \sup_{X \tilde{\times} X} \left| V^\alpha\, \phi^{-\mathfrak{e}} u \right|,$$

where

$$\phi^{-\mathfrak{e}} := \prod_j \phi_j^{-e_j} (\log \phi_j)^{-m_j},$$

for $\mathfrak{e} = \{(e_1, m_1), (e_2, m_2), \dots\}$. The imaginary parts of the e_j are irrelevant in this definition, so the non-uniqueness in the choice of $\mathfrak{e}$ doesn't matter.

The full topology is defined inductively, using C^∞ seminorms on the coefficients $u_{a,l}$ from the expansion (3.3) in local coordinates, and then seminorms of the form $\|\cdot\|_{k;\mathfrak{e}}$ on the remainders when leading terms in the expansion have been subtracted off. In essence, this topology controls all coefficients and all remainders from the asymptotic expansions. With this topology $\mathcal{A}_{\mathcal{E}}(W)$ is also a complete locally convex vector space [20].

Returning now to the lifts of kernels from $X \times X$ to $X \tilde{\times} X$, let $\mathcal{E}$ be an index family for $X \tilde{\times} X$, with the boundary faces ordered left, right, front. We define

$$\tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2}) := \{A : \dot{C}^\infty(X, \Omega^{1/2}) \to \mathcal{D}'(X, \Omega^{1/2});\ \tilde{\mathcal{K}}_A \in \mathcal{A}_{\mathcal{E}}(X \tilde{\times} X;\, \tau\Omega^{1/2})\}.$$

Most commonly, $\mathcal{E}$ will be of the form $\{a, b, c\}$. We also denote

$$\tilde{\Psi}^m_{\mathcal{E}}(X;\, \Omega^{1/2}) = \tilde{\Psi}^m(X;\, \Omega^{1/2}) + \tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2}),$$

with the topology inherited from the two pieces.

Another space we will need consists of operators with smooth kernels on $X \times X$, which are polyhomogeneous conormal at the left and right boundaries. (Such operators have nothing to do with the stretched product.) Given an index family $\mathcal{I}$ for $X \times X$, define

$$\begin{aligned}\Psi_{\mathcal{I}}(X;\, \Omega^{1/2}) := \{A : \dot{C}^\infty(X, \Omega^{1/2}) &\to \mathcal{D}'(X, \Omega^{1/2}); \\ &\mathcal{K}_A \in \mathcal{A}_{\mathcal{I}}(X \times X; (\rho\rho')^{-(n+1)/2}\Omega^{1/2})\}.\end{aligned}$$

The normalization of the half-density bundles is such that

$$\Psi_{a,b}(X;\, \Omega^{1/2}) \subset \tilde{\Psi}_{a,b,a+b}(X;\, \Omega^{1/2}).$$

The composition of operators in $\tilde{\Psi}_{\mathcal{E}}(X;\,\Omega^{1/2})$ combines index families in a straightforward but non-trivial way. Given index sets E_1, E_2, the sum $E_1 + E_2$ has the obvious meaning:

$$E_1 + E_2 = \{(a+b, k+l) :\ (a,k) \in E_1,\ (b,l) \in E_2)\}.$$

But because compositions can introduce extra logarithmic singularities, an extended notion of union is required. If $b \notin a + \mathbb{Z}$, then we simply set

$$\{(a,k)\} \,\overline{\cup}\, \{(b,l)\} := \{(a,k), (b,l)\}.$$

But if $b \in a + \mathbb{N}_0$, then we define

$$\{(a,k)\} \,\overline{\cup}\, \{(b,l)\} := \{(a,k), (b, k+l+1)\}.$$

This notion is extended to full index sets in the obvious way. We also define

$$\operatorname{Re} E := \min\{\operatorname{Re} a : (a,k) \in E\}.$$

The index set $E = \infty$ has the additive property suggested by the notation, but behaves as the empty set in unions:

$$\infty + F = \infty, \qquad \infty \,\overline{\cup}\, F = F.$$

The following result is taken from Theorems 3.15 and 3.18 of Mazzeo's paper (we use a slightly different convention for the index sets).

Theorem 3.1. [13][3] *For $m, k \in \mathbb{Z}$ and index sets given by $\mathcal{E} = \{E_1, E_2, E_3\}$, $\mathcal{F} = \{F_1, F_2, F_3\}$ such that $\operatorname{Re} E_2 + \operatorname{Re} F_1 > n$, composition gives a continuous map*

$$\tilde{\Psi}^m_{\mathcal{E}}(X;\,\Omega^{1/2}) \times \tilde{\Psi}^k_{\mathcal{F}}(X;\,\Omega^{1/2}) \to \tilde{\Psi}^{m+k}_{\mathcal{E}\circ\mathcal{F}}(X;\,\Omega^{1/2}),$$

where

$$\mathcal{E} \circ \mathcal{F} := \{E_1 \,\overline{\cup}\, (F_1 + E_3),\ F_2 \,\overline{\cup}\, (E_2 + F_3),\ (E_3 + F_3) \,\overline{\cup}\, (E_1 + F_2)\}.$$

Also, for $m \in \mathbb{Z}$ and index sets $\mathcal{E} = \{E_1, E_2, E_3\}, \mathcal{I} = \{I_1, I_2\}$ such that $\operatorname{Re} E_2 + \operatorname{Re} I_1 > n$, composition gives a continuous map

$$\tilde{\Psi}^m_{\mathcal{E}}(X;\,\Omega^{1/2}) \times \Psi_{\mathcal{I}}(X;\,\Omega^{1/2}) \to \Psi_{\mathcal{E}\circ\mathcal{I}}(X;\,\Omega^{1/2}),$$

where

$$\mathcal{E} \circ \mathcal{I} := \{E_1 \,\overline{\cup}\, (I_1 + E_3), I_2\}.$$

To compose operators, the respective kernels are pulled back to $X \times X \times X$, multiplied together, and then pushed forward. To keep track of the conormal singularities in the first composition formula above, $X\times X\times X$ is replaced by a blown up version, denoted by $\widetilde{X^3}$. This space is equipped with maps $\beta_{ij} : \widetilde{X^3} \to X \tilde{\times} X$, $i, j = 1, 2, 3$, which correspond on the interior to projections onto the i-th and j-th factors. The crucial feature is that these maps are boundary fibrations in the sense of [20]. The composition of $f, g \in \mathcal{A}_*(X \tilde{\times} X;\, \tau\Omega^{1/2})$ is given by

$$f \circ g := \beta_{13*}[\beta_{12}^* f \cdot \beta_{23}^* g]. \tag{3.4}$$

One can also study the action of $\tilde{\Psi}^m_{\mathcal{E}}(X;\,\Omega^{1/2})$ on conormal functions on X by similar methods. We will need only the following.

[3]The continuity of the compositions was not actually considered explicitly in [13]. The techniques used in [13] are shown to be continuous in [20].

Proposition 3.2. [13] *For* $A \in \tilde{\Psi}^m_{\mathcal{E}}(X;\,\Omega^{1/2})$ *and* $u \in \dot{C}^\infty(X;\Omega^{1/2})$*,* Au *is well-defined and*

$$Au \in \mathcal{A}_{E_1}(X;\rho^{-(n+1)/2}\Omega^{1/2}),$$

where E_1 *is the index set in* $\mathcal{E}$ *corresponding to the left face.*

Given the metric g the natural Hilbert space is $L^2(X,\mu_g^2)$. The map $f \mapsto f\mu_g$ removes the dependency on the metric and defines an isometry $L^2(X,\mathrm{vol}_g) \to L^2(X;\rho^{-(n+1)/2}\Omega^{1/2})$. We will need to consider weighted L^2 spaces of the form $\rho^\delta L^2(X;\rho^{-(n+1)/2}\Omega^{1/2})$ for $\delta > 0$.

Proposition 3.3. [13] *Operators in* $\tilde{\Psi}_{\mathcal{E}}(X;\,\Omega^{1/2})$*, where* $\mathcal{E} = \{E_1,E_2,E_3\}$*, are compact on* $\rho^\delta L^2(X;\rho^{-(n+1)/2}\Omega^{1/2})$ *provided that* $\operatorname{Re} E_1 > \frac{n}{2}+\delta$*,* $\operatorname{Re} E_2 > \frac{n}{2}-\delta$*, and* $\operatorname{Re} E_3 \geq 0$*.*

3.3. **Symbol map.** We come now the main reason for the introduction of the stretched product. Consider the operators $x\partial_{z_j}$ in local coordinates $z=(x,y)$. Let $A \in \mathrm{Diff}^1_0(X;\Omega^{1/2})$ be given locally by

$$A : f(z)\left|\frac{dx\,dy}{x^{n+1}}\right|^{1/2} \mapsto x\frac{\partial}{\partial z_j}f(z)\left|\frac{dx\,dy}{x^{n+1}}\right|^{1/2}.$$

From (3.1) we see that

$$\tilde{\mathcal{K}}_A = t\frac{\partial}{\partial v_j}[\delta(t-1)\delta(u)]\left|\frac{dx\,dy\,dt\,du}{tx^{n+1}}\right|^{1/2},$$

where $v=(t,u)$. Note that $\tilde{\mathcal{K}}_A$ no longer degenerates; it is transversally elliptic to $\tilde{\Lambda}$, all the way down to the front face $\{x=0\}$.

Given a metric g and $\zeta \in \mathbb{C}$, we define the operator $P_\zeta \in \mathrm{Diff}^2_0(X;\Omega^{1/2})$ by

$$P_\zeta(f\mu_g) := \left[(\Delta_g - \zeta(n-\zeta))\,f\right]\mu_g,$$

where $f \in C^\infty(X)$ and μ_g is the Riemannian half-density associated to g (note that μ_g is singular at the boundary).

Proposition 3.4. [14] $\tilde{\mathcal{K}}_{P_\zeta}$ *is uniformly transversally elliptic to* $\tilde{\Lambda}$*.*

Proof. This may be checked in local coordinates $z=(x,y)$ as above, with $x=\rho$. Locally Δ_g takes the form

$$\Delta_g = -\sum_{j,k}\frac{1}{\sqrt{h}}\big(x\frac{\partial}{\partial z_j}\big)h^{jk}\sqrt{h}\big(x\frac{\partial}{\partial z_k}\big) + n\sum_k h^{0k}\big(x\frac{\partial}{\partial z_k}\big), \tag{3.5}$$

where $h_{ij} = x^2 g_{ij}(x,y)$. If $\mu_g = w(x,y)|\frac{dx\,dy}{x^{n+1}}|^{1/2}$, then

$$\tilde{\mathcal{K}}_{\Delta_g} = L[\delta(t-1)\delta(u)]\frac{w(x,y)}{w(\frac{x}{t},y-\frac{tu}{x})}\left|\frac{dx\,dy\,dt\,du}{tx^{n+1}}\right|^{1/2}.$$

where L is the differential operator:

$$L = -\sum_{j,k}h^{jk}\big(t\frac{\partial}{\partial v_j}\big)\big(t\frac{\partial}{\partial v_k}\big) - \sum_{j,k}\frac{x}{\sqrt{h}}\frac{\partial}{\partial z_j}(h^{jk}\sqrt{h})\big(t\frac{\partial}{\partial v_k}\big) + n\sum_k h^{0k}\big(t\frac{\partial}{\partial v_k}\big), \tag{3.6}$$

with $v = (t, u)$ and $h = h(x, y)$. Now as the front face $\{x = 0\}$ is approached, the principal term is $t^2 \sum h^{jk} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_k} \delta(t-1)\delta(u)$. This proves the proposition, since $t = 1$ on $\tilde{\Lambda}$ and h is smoothly extendible across $x = 0$. $\square$

Proposition 3.4 shows that in local coordinates $\tilde{\mathcal{K}}_{\Delta_g}$ looks like the kernel of an elliptic operator which does not degenerate at the boundary. In fact there is a symbol map for elements of $\tilde{\Psi}^m(X;\, \Omega^{1/2})$, given by taking the symbol of the lifted kernel as a conormal distribution on $X \tilde{\times} X$. Proposition 3.4 is then simply the statement that the $\sigma(\Delta_g)$ defined in this way is invertible.

The symbol of a conormal distribution is a half-density on the conormal bundle of the singular set (see [7]). The conormal bundle of $\tilde{\Lambda}$ in $X \tilde{\times} X$ is naturally identified with a bundle $\tilde{T}^*X$ over X, the so-called *compressed cotangent bundle.* Locally this bundle is spanned by differentials of the form $dx/x, dy_j/x$. $\tilde{T}^*X$ carries a natural symplectic form, and thus a natural density. So symbols can be invariantly identified with functions on $\tilde{T}^*X$ (which is why we use operators on half-densities). These functions will lie in the standard symbol spaces $S^m(\tilde{T}^*X)$.

Proposition 3.5. [14] *The symbol map gives an exact sequence:*

$$0 \longrightarrow \tilde{\Psi}^{m-1}(X;\, \Omega^{1/2}) \longrightarrow \tilde{\Psi}^m(X;\, \Omega^{1/2}) \xrightarrow{\sigma} S^m(\tilde{T}^*X)/S^{m-1}(\tilde{T}^*X) \longrightarrow 0,$$

and the symbol is multiplicative:

$$\sigma(AB) = \sigma(A) \cdot \sigma(B).$$

Using Propositions 3.4 and 3.5, we may invert P_ζ symbolically, with an error in $\tilde{\Psi}^{-\infty}(X;\, \Omega^{1/2})$. As noted above, the kernel of such an operator is in general singular at the corner of $X \times X$. So we need a way to refine the parametrix near the front face.

3.4. **Normal operator.** The tool used to accomplish this problem is the normal operator at the front face. Given a point $p \in \partial X$, let $T_p^+X \cong \mathbb{R}^{n+1}_+$ be the (closed) inward half of T_pX. Given $A \in \tilde{\Psi}^m_{\mathcal{E}}(X;\, \Omega^{1/2})$, $p \in \partial X$, the restriction of $\tilde{\mathcal{K}}_A$ to $b^{-1}(p,p) \subset F_f$ may be regarded as an operator on $\Omega^{1/2}(T_p^+X)$ by convolution. This operator is denoted by $N_p(A)$.

It is easiest to define $N_p(A)$ in local coordinates. Let (x, y, t, u) be local coordinates for $X \tilde{\times} X$, with p given by $(0, y_0)$. We also use (x, y) as coordinates for T_p^+X. Then if

$$\tilde{\mathcal{K}}_A = k(x, y, t, u) \left| \frac{dx\, dy\, dt\, du}{tx^{n+1}} \right|^{1/2},$$

the action of the normal operator is

$$N_p(A) : f(x,y) \left| \frac{dx\, dy}{x^{n+1}} \right|^{1/2} \mapsto \int k(0, y_0, t, u) f(\frac{x}{t}, y - \frac{xu}{t}) \frac{dt\, du}{t} \; \cdot \; \left| \frac{dx\, dy}{x^{n+1}} \right|^{1/2},$$

for $f \in C^\infty(T_p^+X)$. Note that

$$N_p(I) = I.$$

For differential operators in $\mathrm{Diff}^*_0(X)$, the definition amounts to "freezing coefficients" at the boundary: if A is given by $Af\mu \mapsto a(x,y)(x\partial_z)^\alpha f\, \mu$ for some non-vanishing $\mu \in \Omega^{1/2}(X)$, then we have

$$N_p(A) : f \left| \frac{dx\, dy}{x^{n+1}} \right|^{1/2} \mapsto a(0, y_0)(x\partial_z)^\alpha f \left| \frac{dx\, dy}{x^{n+1}} \right|^{1/2} \tag{3.7}$$

Let g_p be the (hyperbolic) metric on T_p^+X given by $x^{-2}h|_p$, where $h|_p$ is interpreted as a constant matrix. It follows from (3.5) and (3.7) that

$$N_p(P_\zeta) = \Delta_{g_p} - \zeta(n-\zeta).$$

As a function of p, $N_p(A)$ is clearly smooth in the interior of F_f. In fact, if we think of $N(A)$ as a function on F_f, then N maps $\tilde\Psi_{a,b,0}(X;\,\Omega^{1/2})$ to $\mathcal{A}_{a,b}(F_f)$. Clearly N is continuous in the topology defined above for polyhomogeneous distributions, since it just amounts to reading off the leading asymptotic coefficient at the front face.

Proposition 3.6. [14] *The normal operator gives an exact sequence*

$$0 \longrightarrow \tilde\Psi_{a,b,1}(X;\,\Omega^{1/2}) \longrightarrow \tilde\Psi_{a,b,0}(X;\,\Omega^{1/2}) \xrightarrow{N} \mathcal{A}_{a,b}(F_f) \longrightarrow 0.$$

For $P \in \mathit{Diff}_0^*(X, \Omega^{1/2})$,

$$N_p(P\cdot K) = N_p(P)\circ N_p(K).$$

4. Resolvent and scattering operator

As in §3, let P_ζ be the operator $\Delta_g - \zeta(n-\zeta)$, acting on half-densities through the Riemannian half-density. The resolvent, $R_\zeta := P_\zeta^{-1}$ exists and is bounded for $\operatorname{Re}\zeta$ sufficiently large, because Δ_g is self-adjoint and positive.

Theorem 4.1. [14], [13] *R_ζ may be analytically continued to a meromorphic family on the domain $\zeta \in \mathbb{C}\backslash\{\frac{1}{2}(n-\mathbb{N})\}$ with*

$$R_\zeta \in \tilde\Psi^{-2}_{\zeta,\zeta,0}(X;\,\Omega^{1/2}) + \Psi_{\zeta,\zeta}(X;\,\Omega^{1/2}).$$

The method of proof is as follows. For $\zeta \notin \frac{1}{2}(n-\mathbb{N})$ a parametrix

$$M_\zeta \in \tilde\Psi^{-2}_{\zeta,\zeta,0}(X;\,\Omega^{1/2})$$

is constructed, such that $P_\zeta M_\zeta - I = E_\zeta \in \Psi_{\infty,\zeta}(X;\,\Omega^{1/2})$. This error term has two crucial features. First of all, it's compact on $\rho^\delta L^2(X, \rho^{-(n+1)/2}\Omega^{1/2})$ for $\delta > 0$, so $I + E_\zeta$ can be inverted meromorphically by analytic Fredholm theory. Secondly, if we define F_ζ by setting $I + F_\zeta = (I+E_\zeta)^{-1}$, then F_ζ is also in $\Psi_{\infty,\zeta}(X;\,\Omega^{1/2})$. This in turn implies that $R_\zeta = M_\zeta + M_\zeta F_\zeta \in \tilde\Psi^{-2}_{\zeta,\zeta,0}(X;\,\Omega^{1/2}) + \Psi_{\zeta,\zeta}(X;\,\Omega^{1/2})$.

We turn next to some facts concerning solutions of $\Delta_g u = \zeta(n-\zeta)u$. The first is essentially a uniqueness result. Although well-known, it doesn't seem to be proven in the literature so we will give a proof. Let $\operatorname{Spec}(\Delta_g) \subset (0, n^2/4)$ be the point spectrum of the Laplacian.

Proposition 4.2. *Assume $\operatorname{Re}\zeta \geq n/2$, $\zeta \neq n/2$ and $\zeta(n-\zeta) \notin \operatorname{Spec}(\Delta_g)$. Suppose that the function u that solves $\Delta_g u = \zeta(n-\zeta)u$ and has asymptotic behavior*

$$u = \rho^\zeta f + O(\rho^{\zeta+\epsilon}), \tag{4.1}$$

for $f \in C^\infty(\partial X)$, with $\epsilon > 0$. Then $u = 0$.

Proof. For $\operatorname{Re}\zeta > n/2$ the expansion (4.1) implies that u is in $L^2(X, \operatorname{vol}_g)$ so in this case the statement is tautological.

For the rest of the proof we assume $\operatorname{Re}\zeta = n/2$. We will show that $\Delta_g u = \zeta(n-\zeta)u$ and (4.1) together imply $f = 0$. This will imply $u \in L^2(X, \operatorname{vol}_g)$, and we conclude $u = 0$ as before.

We use a version of the boundary pairing argument of [17]. Suppose that $\Delta_g u = \zeta(n-\zeta)u$. Let $\psi \in C^\infty(\mathbb{R}_+)$ be a cutoff function so that $\psi(t) = 0$ for $t \le 1$ and $\psi(t) = 1$ for $t \ge 2$. By the self-adjointness of $\Delta_g - \zeta(n-\zeta)$, we have

$$\int_X \Big([\Delta_g, \psi(\lambda\rho)]u \Big)\, \overline{u}\, dg = 0.$$

So in particular

$$\lim_{\lambda\to\infty} \int_X \Big([\Delta_g, \psi(\lambda\rho)]u \Big)\, \overline{u}\, dg = 0. \tag{4.2}$$

In local coordinates (x, y) with $x = \rho$,

$$\Delta_g = -(x\partial_x)^2 + nx\partial_x + p(x, y, x\partial_y) + xT,$$

where $T \in \mathrm{Diff}_0^2(X)$. Thus if u satisfies (4.1)

$$[\Delta_g, \psi(\lambda x)]u = \Big[(n-2\zeta-1)\lambda\psi'(\lambda x) - \lambda^2 x\psi''(\lambda x)\Big] x^{\zeta+1} f + \lambda\psi'(\lambda x) \times O(x^{n/2+1+\epsilon}).$$

Substitute this expression back into (4.2). After we perform the x integration, only the term involving f will survive as $\lambda \to \infty$ (recall that $dg = dh/x^{n+1}$). The conclusion is that

$$0 = (n-2\zeta) \int_{\partial X} |f|^2\, dh|_{\partial X},$$

so f must vanish. ☐

The second result describes the asymptotic behavior of solutions of $\Delta_g u = \zeta(n-\zeta)u$ and allows us to define the scattering operator. Once again this is well-known, and more general results on asymptotic expansions of generalized eigenfunctions may be found in [13].

Proposition 4.3. *Let* $\operatorname{Re}\zeta = n/2, \operatorname{Im}\zeta \neq 0$. *Given* $f \in C^\infty(\partial X)$ *there is a unique solution of* $\Delta_g u = \zeta(n-\zeta)u$ *which near* ∂X *has the asymptotic form:*

$$u(z) = \rho^{n-\zeta} f + \rho^{\zeta} f' + O(\rho^{n/2+\epsilon}).$$

Proof. As in the model case, we define

$$E_\zeta(q, p') = \lim_{q'\to p'} \rho(q')^{-\zeta} \frac{\mathcal{K}_{R_\zeta}(q, q')}{\mu_g \mu'_g}, \tag{4.3}$$

and note that

$$u(q) = 2^{2\zeta}(2\zeta - n) \int E_\zeta(q, p') f(p')\, dh|_{\partial X}$$

is a solution of $\Delta_g u = \zeta(n-\zeta)u$. Choose local coordinates (x, y) around $p = (0, y_0)$ for which g_p is the standard hyperbolic metric. Then because $N_p(\Delta_g) = \Delta_{g_p}$ and by the exact sequence of Proposition 3.6, in the coordinates $(\eta, \eta', \theta, r, y_0)$ we have

(4.4)

$$\tilde{\mathcal{K}}_{R_\zeta}(\eta, \eta', \theta, r, y_0) = \Big(G_\zeta(\eta, \eta', \theta) + r(\eta\eta')^\zeta F(\eta, \eta', \theta, r, y_0) \Big) \left| \frac{d\eta\, d\eta'\, d\theta\, dr\, dy}{(\eta\eta' r)^{n+1}} \right|^{1/2},$$

where F is smooth. Here G_ζ is the model resolvent, which depends only on η, η', and θ because in these coordinates the hyperbolic distance d is given by

$$\cosh d = 1 + \frac{|\eta - \eta'|^2 + \theta^2}{2\eta\eta'}.$$

From (4.4) and (4.3) we compute that

$$E_\zeta(x, y_0, y') = \frac{c_\zeta x^\zeta}{r^{2\zeta}} + \frac{x^\zeta}{r^{2\zeta-1}} F(x/r, 0, (y_0 - y')/r, r, y_0), \tag{4.5}$$

where $r = \sqrt{x^2 + (y_0 - y')^2}$ and c_ζ is the constant (2.1). The analysis proceeds as in Proposition 2.1.

Uniqueness of the solution follows immediately from Proposition 4.2. □

Using Proposition 4.3, for $\operatorname{Re}\zeta = n/2$ we define S_ζ to be the operator which maps $f \mapsto f'$. From the proof of the proposition it is clear that the Schwartz kernel with respect to the Riemannian density on ∂X induced by h is

$$K_{S_\zeta}(p, p') = 2^{2\zeta}(2\zeta - n) \lim_{z \to p} \lim_{z' \to p'} \rho(z)^{-\zeta} \frac{\mathcal{K}_{R_\zeta}(z, z')}{\mu_g \mu'_g} \rho(z')^{-\zeta}, \tag{4.6}$$

for $p \neq p' \in \partial X$. We can adopt (4.6) more generally to define S_ζ meromorphically in ζ.[4] Note that Proposition 4.3 implies that for $\operatorname{Re}\zeta = n/2$,

$$S_{n-\zeta} S_\zeta = I.$$

Since $S(\zeta)$ is a meromorphic family, this relation continuous to hold for all $\zeta \notin \frac{1}{2}(n - \mathbb{N})$.

From (4.6) and the form of $\tilde{\mathcal{K}}_{R_\zeta}$ we see that in local coordinates (y, y') the kernel of the scattering operator has the form

$$\mathcal{K}_{S_\zeta} = r^{-2\zeta} F(r, \theta, y) + G(y, y'), \tag{4.7}$$

where $r = |y - y'|$, $\theta = (y - y')/r$, and F and G are C^∞ in their respective variables. Let $\Psi^a(\partial X)$ be the set of one-step pseudifferential operators of order $a \in \mathbb{C}$. That is, local Fourier transforms near the diagonal give symbols with full asymptotic expansions of the form

$$a(x, \xi) \sim \sum_{j=0}^{\infty} a_j(x, \xi) |\xi|^{a-j},$$

where the a_j's are all homogeneous of degree zero. The topology on $\Psi^a(\partial X)$ is given by applying the seminorms from the symbol class S^0 to each coefficient a_j in the local expansions near the diagonal (together with C^∞ seminorms away from the diagonal).

Proposition 4.4.

$$S_\zeta \in \Psi^{2\zeta - n}(\partial X).$$

The principal symbol of S_ζ is $(|\xi|_{h|\partial X})^{2\zeta - n}$ times a function meromorphic in ζ and independent of g.

Proof. The first statement follows immediately from the local form (4.7). The existence of the one-step expansion corresponds to the smoothness of F as a function of r. The principal symbol of S_ζ is derived directly from the limiting form for E_ζ given in local coordinates by (4.5). □

There are standard relations between R_ζ, E_ζ, and S_ζ. For proofs of the following, see for example Theorems 5.3 and 6.3 of [22].

[4] Note that this limiting process will introduce new poles in S_ζ which were not poles of R_ζ, because $\mathcal{K}_{S_\zeta}$ must be interpreted as a distribution. This issue of resolvent vs. scattering poles will be dealt with in [3].

Proposition 4.5.

$$\mathcal{K}_{R_{n-\zeta}}(q,q') = \mathcal{K}_{R_\zeta}(q,q') + (n-2\zeta)\mu_g\mu'_g \int_{\partial X} E_\zeta(q,p)E_{n-\zeta}(q',p)\, dh|_{\partial X},$$

and

$$E_{n-\zeta}(q,\cdot) = S_{n-\zeta}E_\zeta(q,\cdot).$$

5. Continuity

The construction of the parametrix M_ζ in [14] involves summing three asymptotic series, at the lifted diagonal $\tilde{\Lambda}$, the front face, and the left face. These summations could each be performed so as to insure continuity in g, but it turns out that we can use an abbreviated construction with only the first summation. The reason is that in the final stage of our argument we will need to use a uniqueness property of the resolvent to extend continuity from the parametrix to the resolvent. The uniqueness, which comes from Proposition 4.2, is strong enough to apply even with a cruder parametrix. The error term for our parametrix will lie in $\tilde{\Psi}_{\zeta+1,\zeta,1}(X;\Omega^{1/2})$ rather than $\Psi_{\infty,\zeta}(X;\,\Omega^{1/2})$.

Let $\mathcal{M}_X$ denote the space of smooth, asymptotically hyperbolic metrics on X. The C^∞ topology on $\mathcal{M}_X$ is defined by seminorms of the form

$$\|\omega\| = \sup_X |V_1 \dots V_m \rho^2 \omega(Y,Z)|,$$

where $V_1,\dots,V_m,Y,Z \in \mathcal{V}(X)$.

As a preliminary, we note the following, which follows easily from the definitions of the topologies.

Lemma 5.1. *For any $\zeta \in \mathbb{C}$, the map*

$$\mathcal{M}_X \ni g \mapsto P_\zeta \in \tilde{\Psi}^2(X;\,\Omega^{1/2})$$

is continuous.

Our main goal in this section is to extend this continuity from P_ζ to its inverse.

5.1. The half-plane $\operatorname{Re}\zeta \geq n/2$.

Theorem 5.2. *Assume $\operatorname{Re}\zeta \geq \frac{n}{2}$, $\zeta \neq n/2$. The map*

$$\mathcal{M}_X \ni g \mapsto R_\zeta \in \tilde{\Psi}^{-2}_{\zeta,\zeta,0}(X;\,\Omega^{1/2}) + \Psi_{\zeta,\zeta}(X;\,\Omega^{1/2})$$

is continuous, except at points where $\zeta(n-\zeta) \in \operatorname{Spec}(\Delta_g)$.

We begin with an elementary topological lemma.

Lemma 5.3. *Let W, d be a separable metric vector space with a dense vector subspace W_0. Suppose F is a continuous map from a topological space $Y \to W$. Then for any $\epsilon > 0$ there exists a continuous map $G: Y \to W_0$ such that $d(F(y), G(y)) < \epsilon$ for all $y \in Y$.*

Proof. Since W is separable and W_0 is dense, we may choose a countable set $w_i \in W_0$ such that $W = \cup_i B_\epsilon(w_j)$, where $B_\epsilon(w_j) = \{w \in W : d(w,w_j) < \epsilon\}$. Let ϕ_j be a positive continuous function such that $0 \leq \phi_j(w) \leq \min\{2^{-j}, 2^{-j}/d(0,w_j)\}$ and $\operatorname{supp}\phi_j = \overline{B_\epsilon(w_j)}$. Then $\phi = \sum_j \phi_j$ converges uniformly on W and hence is continuous. Also $\phi(w) \neq 0$ for all $w \in W$. Note that $\sum_j \phi_j(F(y))w_j$ converges uniformly on Y and so defines a continuous function $Y \to W$. Then $G(y) = \frac{1}{\phi}\sum_j \phi_j(F(y))w_j$ has the desired properties. ∎

Proof of Theorem 5.2. For notational convenience, choose a metric d_m for each $\tilde{\Psi}^m(X;\,\Omega^{1/2})$, such that $d_m \leq d_{m-1}$.

The first stage of the construction is to remove the conormal singularity at $\tilde{\Lambda}$; this is just a standard parametrix construction using the symbol map. To start, let $Q_{0,0} = I$. At each inductive step, we are given $Q_{0,j} \in \tilde{\Psi}^{-j}(X;\,\Omega^{1/2})$ which is continuous in g with respect to d_{-j}. We choose A_j with symbol

$$\sigma(A_j) = \sigma(Q_{0,j})/\sigma(P_\zeta).$$

Clearly we may do this so that A_j is a continuous function of g in $\tilde{\Psi}^{-2-j}(X;\,\Omega^{1/2})$. Then, using Lemma 5.3, we may find $E_j \in \tilde{\Psi}^{-\infty}(X;\,\Omega^{1/2})$ such that

$$d_{-2-j}(\tilde{\mathcal{K}}_{A_j}, \tilde{\mathcal{K}}_{E_j}) < 2^{-j}$$

and E_j is continuous in as an element of $\tilde{\Psi}^{-2-j}(X;\,\Omega^{1/2})$. We take

$$Q_{0,j+1} = Q_{0,j} - P_\zeta(A_j - E_j) \in \tilde{\Psi}^{-j-1}(X;\,\Omega^{1/2}),$$

also continuous, and proceed to the next step.

Finally, we set

$$A = \sum_{j=0}^{\infty} (A_j - E_j),$$

which converges uniformly in $\tilde{\Psi}^{-2}(X;\,\Omega^{1/2})$ and so is continuous as a function of g. Let

$$Q_1 = I - P_\zeta A \in \tilde{\Psi}^{-\infty}(X;\,\Omega^{1/2})$$

For any N we can write

$$Q_1 = Q_{0,N} - \sum_{j=N+1}^{\infty} P_\zeta(A_j - E_j),$$

which shows that Q_1 is continuous with respect to d_{-N}. Thus Q_1 is a continuous function of g in the topology of $\tilde{\Psi}^{-\infty}(X;\,\Omega^{1/2})$.

The next stage is to use the model resolvent on $T_p^+X \cong \mathbb{H}^{n+1}$ to find $B \in \tilde{\Psi}_{\zeta,\zeta,0}(X;\,\Omega^{1/2})$ which solves

$$[\Delta_{g_p} - \zeta(n-\zeta)] \cdot N_p(B) = N_p(Q_1), \tag{5.1}$$

for each $p \in F_f$. By the exact sequence of Proposition 3.6, we then have

$$P_\zeta B + Q_1 = Q_2 \in \tilde{\Psi}_{\zeta,\zeta,1}(X;\,\Omega^{1/2}). \tag{5.2}$$

Note however, that for any $f \in C^\infty(X)$,

$$(\Delta_g - \zeta(n-\zeta))\rho^\zeta f = O(\rho^{\zeta+1}).$$

Since $\tilde{\mathcal{K}}_B$ has the form $\phi_l^\zeta \times$ (smooth) near the left face, and Q_1 vanishes there to infinite order, we see from (5.2) that in fact

$$Q_2 \in \tilde{\Psi}_{\zeta+1,\zeta,1}(X;\,\Omega^{1/2})$$

It is not difficult to see that this step is continuous. Consider the solution of (5.1). By linear change of coordinates, assume g_p is the standard hyperbolic metric. The space T_p^+X, on which $N_p(\cdot)$ lives, is diffeomorphic to the interior of the fiber S_{++}^{n+1}. This fiber has boundary defining functions σ_l and σ_r, which are the restrictions of

ϕ_l and ϕ_r. Given $Q_1 \in \tilde{\Psi}^{-\infty}(X;\, \Omega^{1/2})$, $N_p(Q_1)$ defines an element of $\dot{C}^\infty(S^{n+1}_{++})$. We have already noted that the map

$$\tilde{\mathcal{K}}_{Q_1} \mapsto N_p(Q_1).$$

is continuous. Lemma 6.13 of [14] says that

$$G_\zeta : \dot{C}^\infty(S^{n+1}_{++}) \to \sigma_l^\zeta \sigma_r^\zeta C^\infty(S^{n+1}_{++}),$$

is continuous. This gives us a function on F_f which we may clearly extend into $X \tilde{\times} X$ so as to preserve the continuity. The conclusion is that the map $\tilde{\mathcal{K}}_{Q_1} \mapsto \tilde{\mathcal{K}}_B$ giving to the solution of (5.1) may be made continuous as a map

$$\mathcal{A}_{\infty,\infty,0}(X \tilde{\times} X;\, \tau\Omega^{1/2}) \to \mathcal{A}_{\zeta,\zeta,0}(X \tilde{\times} X;\, \tau\Omega^{1/2}).$$

Since $Q_2 = P_\zeta B + Q_1$, it follows from Theorem 3.1 that $\tilde{\mathcal{K}}_{Q_2}$ also depends continuously on Q_1 and P_ζ, as an element of $\mathcal{A}_{\zeta+1,\zeta,1}(X \tilde{\times} X; \tau\Omega^{1/2})$. At this stage we have $P_\zeta(A+B) = I - Q_2$, where the maps $g \mapsto \tilde{\mathcal{K}}_A$, $\tilde{\mathcal{K}}_B$ and $\tilde{\mathcal{K}}_Q$ may be assumed continuous in $I^{-2}(X \tilde{\times} X, \tilde{\Lambda}; \tau\Omega^{1/2})$, $\mathcal{A}_{\zeta,\zeta,0}(X \tilde{\times} X; \tau\Omega^{1/2})$, and $\mathcal{A}_{\zeta+1,\zeta,1}(X \tilde{\times} X; \tau\Omega^{1/2})$, respectively

The remaining task is to invert $I - Q_2$. The operator Q_2 is compact on an appropriate weighted L^2 space, but the Neumann series for $(I-Q_2)^{-1}$ doesn't necessarily converge. In order to make use of the Neumann series, we fix a particular metric g_0 and consider only metrics in a neighborhood of g_0 in $\mathcal{M}_X$. By adding a term $C \in \Psi_{\zeta,\zeta}(X;\, \Omega^{1/2})$, and adjusting the earlier construction of A and B accordingly, we may assume that $R_\zeta(g_0) = (A + B + C)(g_0)$. Of course we also assume that C depends continuously on g. Thus, given seminorms on the appropriate spaces, we may construct A, B, C, Q such that

$$P_\zeta(A + B + C) = I - Q,$$

with all operators are continuous functions of g in the appropriate spaces. And in addition, $Q(g_0) = 0$.

Lemma 5.4. *Let $\mathcal{E}$ be an index family such that $E_1 + E_2 > n$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. The topology on $\tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2})$ may be defined with a family of seminorms $\{\|\cdot\|_a\}$, which each have the property that*

$$\|A \circ B\|_a \leq \|A\|_a \, \|B\|_a, \tag{5.3}$$

for any $A, B \in \tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2})$.

Proof. By the continuity of the composition,

$$\tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2}) \circ \tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2}) \to \tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2}),$$

we can always estimate a seminorm on the right-hand side with some different seminorms on the left-hand side. The point we need to check is that the estimates on the left will require no more derivative than those on the right. To estimate the derivative of $\tilde{\mathcal{K}}_{A\circ B}$ by some vector field $V \in \mathcal{V}_b(X \tilde{\times} X)$, recall how the composition is defined in (3.4). We lift V to an element of $\mathcal{V}_b(\widetilde{X^3})$ and apply it to $\beta_{12}^* \tilde{\mathcal{K}}_A \cdot \beta_{23}^* \tilde{\mathcal{K}}_B$. Clearly the result may be estimated by some combination of bounds on elements of $\mathcal{V}_b(X \tilde{\times} X)$ applied to $\tilde{\mathcal{K}}_A$ and $\tilde{\mathcal{K}}_B$, together with sup-norms of each. (Of course the appropriate weightings at the boundary must be included.) So if $\|A\|_a$ includes estimates the derivatives of $\tilde{\mathcal{K}}_A$ under a spanning set of vector fields in $\mathcal{V}_b(X \tilde{\times} X)$

(plus estimates of the undifferentiated $\tilde{\mathcal{K}}_A$), then we have (5.3) up to a constant. The constant is removed by rescaling.

By induction, we can do the same for seminorms with arbitrary numbers of derivatives. (Our seminorms will all be norms, in fact.)

This would suffice for the conormal topology. The polyhomogeneous conormal topology requires in addition estimates on all coefficients in boundary expansions. But the argument is essentially the same. We remove the leading terms in the boundary expansions (in order to study the remainders) by applying particular differential operators. For example, in local coordinates (x, y), applying $x\partial_x - \alpha$ to $x^\alpha f$ yields $x^\alpha \partial_x f$, whose leading term is the second coefficient in the original expansion. (Such operators are often used to define polyhomogeneity, as in [16].) So estimates of boundary coefficients of $\tilde{\mathcal{K}}_{A\circ B}$ may be done by lifting such operators to $\widetilde{X^3}$, and a similar argument applies. □

Formally, $(I - Q)^{-1} = I + Q'$, where

$$Q' = \sum_{j=1}^{\infty} Q^j.$$

Assuming that this series converges, the resulting operator will be an element of $\tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2})$, where

$$E_1 = \{(\zeta + 1 + k, k);\ k = 0, 1, \dots\},$$
$$E_2 = \{(\zeta + k, k);\ k = 0, 1, \dots\},$$
$$E_3 = \{1\} \cup \{(2\zeta + k, (k^2 + k - 2)/2);\ k = 1, 2, \dots\}$$

Fix a seminorm $\|\cdot\|_a$ on $\tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2})$ with the property given in Lemma 5.4.. Then we define the neighborhood

$$W_a = \{g \in \mathcal{M}_X : \|Q\|_a < 1/2\}.$$

The series for Q' converges in $\|\cdot\|_a$, uniformly for $g \in W_a$. So in this neighborhood we have a well-defined Q' which depends continuously on g as measured by $\|\cdot\|_a$. We may assume that the seminorm is strong enough to guarantee that Q' is well-defined as an operator on $\rho^\delta L^2(X; \rho^{-(n+1)/2}\Omega^{1/2})$.

Now let $M = (A+B+C)(I+Q')$. By construction, M is a right inverse of P_ζ on $\rho^\delta L^2(X; \rho^{-(n+1)/2}\Omega^{1/2})$ for $g \in W_a$ and is a continuous function of g with respect to some particular seminorm (related to $\|\cdot\|_a$) on $\tilde{\Psi}^{-2}_{\zeta,\zeta,0}(X; \Omega^{1/2}) + \tilde{\Psi}_{(\zeta,\zeta,0)\overline{\cup}\mathcal{E}}(X; \Omega^{1/2})$. We will show $M = R_\zeta$.

This will follow from the uniqueness result proved in Proposition 4.2, by the following argument. For $u \in \dot{C}^\infty(X; \Omega^{1/2})$, Mu is certainly well-defined. If Q' actually converged in $\tilde{\Psi}_{\mathcal{E}}(X;\, \Omega^{1/2})$, then Theorem 3.1 and Proposition 3.2 would imply that

$$Mu \in \mathcal{A}_F(X; \rho^{-(n+1)/2}\Omega^{1/2}),$$

where the index set $F = \{(\zeta + k, k); k = 0, 1, \dots\}$. We can't quite assume this, but by making $\|\cdot\|_a$ strong enough, we can at least insure that

$$Mu = [\rho^\zeta f + O(\rho^{\zeta+\epsilon})]\ \mu_g.$$

We also know $P_\zeta(Mu - R_\zeta u) = 0$. The hypotheses on ζ allow us to apply Proposition 4.2 and conclude that $Mu = R_\zeta u$. And since $\dot{C}^\infty(X; \Omega^{1/2})$ is dense in any space we would care to consider, this means $M = R_\zeta$.

Since g_0 and $\|\cdot\|_a$ were arbitrary, this means we can control the continuity of R_ζ in the topology of $\tilde{\Psi}^{-2}_{\zeta,\zeta,0}(X;\Omega^{1/2})+\tilde{\Psi}_{(\zeta,\zeta,0)\overline{\cup}\mathcal{E}}(X;\Omega^{1/2})$. Since the topology on polyhomogeneous conormal distributions includes control of all asymptotic coefficients, this implies continuity in the smaller space $\tilde{\Psi}^{-2}_{\zeta,\zeta,0}(X;\,\Omega^{1/2})+\Psi_{\zeta,\zeta}(X;\,\Omega^{1/2})$ □

5.2. **Extension to the whole plane.** From Theorem 5.2 we easily deduce the following.

Corollary 5.5. *Assume* $\operatorname{Re}\zeta \geq \frac{n}{2}$, $\zeta \neq n/2$. *The map*

$$\mathcal{M}_X \ni g \mapsto S_\zeta \in \Psi^{2\zeta-n}(\partial X)$$

is continuous, except at poles

Proof. The kernel of the scattering operator has the local form

$$K_{S_\zeta}(y,y') = r^{-2\zeta}F(r,\theta,y)+G(y,y'),$$

where (y,y') are local coordinates for $\partial X\times\partial X$, $r=|y-y'|$, $\theta=(y-y')/r$, and F and G are smooth in their respective variables. These functions F and G are just coefficients in the boundary expansion of $\tilde{\mathcal{K}}_{R_\zeta}$, so Theorem 5.2 immediately yields the continuity of the maps $g\mapsto F,G$ in a C^∞ topology. This in turn yields the continuity of S_ζ in the topology of $\Psi^{2\zeta-n}(\partial X)$, except at values of ζ for which the distribution $r^{-2\zeta}$ has a pole. □

This result may be extended to general ζ using the relation[5]

$$S_{n-\zeta}S_\zeta = I. \tag{5.4}$$

We also need the fact that compositions

$$\Psi^a(\partial X)\circ\Psi^b(\partial X)\to\Psi^{a+b}(\partial X)$$

are continuous, which is standard.

Proof of Theorem 1.2. Pick a metric g_0 and suppose that $\operatorname{Re}\zeta > n/2$ and $S_\zeta(g_0)$ exists and is invertible. So there is no scattering pole at either ζ or $n-\zeta$ for g_0. That $S_\zeta(g)$ is also well-defined in a neighborhood of g_0 follows from Corollary 5.5. We will show that $S_\zeta(g)$ is also invertible for metrics in a neighborhood of g_0 and that this inverse is continuous.

Let $T_\zeta(g)\in\Psi^{n-2\zeta}(\partial X)$ be a family of pseudodifferential parametrices for $S_\zeta(g)$ which are continuous functions of g in some neighborhood of g_0. Such a family may be constructed as in the first phase of the proof of Theorem 5.2. Thus we can assume

$$S_\zeta T_\zeta = I + K_\zeta,$$

where $g\mapsto K_\zeta\in\Psi^{-\infty}(\partial X)$ is also continuous. Furthermore, we can arrange that $T_\zeta(g_0)=S_{n-\zeta}(g_0)$, so $K_\zeta(g_0)=0$. From the identity (5.4), we see that $S_{n-\zeta}$ is given by

$$S_{n-\zeta} = T_\zeta(I+K_\zeta)^{-1},$$

when this inverse exists. By the same methods as in the final phase of the proof of Theorem 5.2, $I+K_\zeta$ must be invertible in a neighborhood of g_0 and will depend continuously on g. □

Finally, using Theorem 1.2 we extend Theorem 5.2 to the whole plane.

[5]This idea was pointed out to me by Peter Perry.

Proof of Theorem 1.1. From Proposition 4.5 we have the relation:

(5.5)
$$\mathcal{K}_{R_{n-\zeta}}(q,q') = \mathcal{K}_{R_\zeta}(q,q') + (n-2\zeta)\mu_g\mu_g' \int_{\partial X} E_\zeta(q,p)\ [S_{n-\zeta}E_\zeta](q',p)\, dh|_{\partial X}.$$

E_ζ is a polyhomogeneous conormal distribution on $X \tilde{\times} \partial X$, the stretched version of $X \times \partial X$ defined by local polar coordinates $r = \sqrt{x^2 + |y-y'|^2}$ and $(\eta, \theta) = (x, y-y')/r$. It's continuity as a function of g follows immediately from Theorem 5.2 for $\operatorname{Re}\zeta \geq n/2$.

So for $\operatorname{Re}\zeta \geq n/2$ (and away from poles of $S_{n-\zeta}$), we have established the continuity of all kernels appearing on the right-hand side of (5.5). The final step is to write the integral over ∂X as a sequence of pull-backs and push-forwards, and apply the machinery of [20]. We omit the details. □

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